Two-qubit mixed states more entangled than pure states

Comparison of the relative entropy of entanglement for a given nonlocality

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CHSH inequality violation (B – "Nonlocality")

The two-qubit Bell inequality in the form derived by Clauser, Horne, Shimony and Holt (CHSH) [1] can be formulated as

$$|\mathrm{Tr}\,(\rho\,\mathcal{B}_{\mathrm{CHSH}})| \leq 2,$$

where the Bell-CHSH operator is $\mathcal{B}_{\mathrm{CHSH}}$ is given by

$$\mathcal{B}_{\mathrm{CHSH}} = \vec{a}\cdot\vec{\sigma}\otimes(\vec{b}+\vec{b}')\cdot\vec{\sigma} + \vec{a}'\cdot\vec{\sigma}\otimes(\vec{b}-\vec{b}')\cdot\vec{\sigma},$$

and its expected value is maximized over real-valued three-dimensional unit vectors $\vec{a}$, $\vec{a}'$, $\vec{b}$, and $\vec{b}'$. According to the Horodecki theorem [2,3], the maximum expected value of the Bell-CHSH operator for a given state ρ reads as:

$$\max_{\mathcal{B}_{\mathrm{CHSH}}} \mathrm{Tr}\,(\rho\,\mathcal{B}_{\mathrm{CHSH}}) = 2\sqrt{M(\rho)}$$

given in terms of the parameter

$$M(\rho) = \max_{j<k}\{h_j + h_k\} \leq 2,$$

where h_j ($j = 1,2,3$) are the eigenvalues of the real symmetric matrix $U = T^T T$ constructed from the correlation matrix T and its transpose T^T. Hence, the condition for violating the Bell-CHSH inequality is $M(\rho) > 1$. For convenience, we refer to *nonlocality* as the violation of the Bell-CHSH inequality.

To quantify a degree of the nonlocality, one can use quantity denoted by $B(\rho)$, which for two-qubit pure states is equal to the concurrence and negativity. This measure can be given as:

$$B(\rho) \equiv \sqrt{\max[0,\, M(\rho) - 1]}.$$

We see that $B(\rho) = 0$ if a given state ρ satisfies the Bell-CHSH inequality, and $B(\rho) = 1$ if the inequality is maximally violated. We refer to this BIV degree as *nonlocality* measure.

Relative entropy of entanglement (E_R – REE)

In our considerations, the most important entanglement measure [4] is the REE defined as

$$E_R(\rho) = \min_{\sigma\in\mathcal{D}} S(\rho||\sigma),$$

which is the relative entropy $S(\rho||\sigma) = \mathrm{Tr}\,(\rho\log_2\rho - \rho\log_2\sigma)$ minimized over a set $\mathcal{D}$ of separable states σ. The REE has a desirable property of a good entanglement measure that, for pure states, it reduces to the von Neumann entropy of one of the subsystems. There is no closed general formula for REE [5].

Negativity (N)

The second measure studied here is the negativity, defined as

$$N(\rho) = \max\{0, -2\mu_{\min}\},$$

where $\mu_{\min} = \min\mathrm{eig}(\rho^{\Gamma})$ and Γ denotes partial transpose. The negativity is related to the logarithmic negativity, $\log_2[N(\rho)+1]$, which is a measure of the entanglement cost under the PPT operations. These two related measures reach unity for the Bell states and vanish for separable states.

Concurrence (C)

The last entanglement measure applied in this paper is the concurrence introduced by Wootters as

$$C(\rho) = \max\{0, 2\lambda_{\max} - \sum_j \lambda_j\},$$

where $\lambda_j^2 = \mathrm{eig}[\rho(\sigma_2\otimes\sigma_2)\rho^*(\rho_2\otimes\sigma_2)]_j$ and $\lambda_{\max} = \max_j \lambda_j$. This measure is directly related to the entanglement of formation, $E_F(\rho)$.

Conclusions

We studied the relation between the relative entropy of entanglement E_R and the nonlocality measure B corresponding to a degree of the violation of the Bell-CHSH inequality in two-qubit systems (for details see Ref. [6]). We found states of the extremal value of E_R for a given value of B. We showed that the obtained states satisfy the Karush-Kuhn-Tucker conditions, as well as they provide a boundary for the REE values obtained by our Monte Carlo simulations. We demonstrated that mixed states can be more entangled in terms of E_R than pure states if the nonlocality $B \in (0, 0.82)$ and the maximal difference between these REEs is $\Delta E_R = 0.4$.

We showed that these maximally-entangled mixed states can be obtained from pure two-qubit entangled states by locally subjecting one or both qubits (of the entangled pair) to the amplitude-damping channel either. While the minimally-entangled states can be generated from pure states by subjecting one or both qubits to the phase-damping channel. We found that the amplitude-damped states are more entangled than the phase-damped states (blue and green areas) for a given nonlocality $B < 0.8169$ (corresponding to the nonlocality of the states ρ_4 and ρ_5).

However, for values $B > 0.5856$ (point ρ_6) there exists a range of states, obtained either by the amplitude or phase damping, which have the same value of E_R for a given B. Moreover, we found that the upper bound on the REE of the phase-damped states for a given nonlocality is provided by pure states. However, pure states have the highest REE only for $B > 0.8169$.

Thus, we found that for a large range of the nonlocality, mixed states can be more entangled that pure states, and, surprisingly, these mixed states can by obtained by the ordinary amplitude damping of pure states.

We believe that these results can stimulate further quest for practical protocols of quantum information processing for which mixed states are more effective than pure states.

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Pure states (P)

In this section we analyze the effect of the prototype damping channels on pure states of the form

$$|\psi_\alpha\rangle = \sqrt{\alpha}|01\rangle + \sqrt{1-\alpha}|10\rangle,$$

where $0 \leq \alpha \leq 1$. When analyzing the entanglement measures, any pure two-qubit state, can be equivalently expressed by means of the Schmidt decomposition, in the form given above.

Kraus operator framework

A convenient description of damping can be given in terms of the Kraus operators E_i (specified below). Two-side damping of the state $\rho_{\mathrm{in}}(\alpha) = |\psi_\alpha\rangle\langle\psi_\alpha|$ leads to the following output state

$$\rho(\alpha, q_1, q_2) = \sum_{i,j}[E_i(q_1)\otimes E_j(q_2)]\rho_{\mathrm{in}}(\alpha)[E_i^\dagger(q_1)\otimes E_j^\dagger(q_2)],$$

where the Kraus operators satisfy the normalization relation $\sum_i E_i^\dagger(q)E_i(q) = I$. In the special case of the one-side damping (say $q_2 = 0$), reduces to

$$\rho(\alpha, q_1) = \sum_i [E_i(q_1)\otimes I]\rho_{\mathrm{in}}(\alpha)[E_i^\dagger(q_1)\otimes I].$$

Amplitude damping channel (ADC)

The Kraus operators for the single-qubit amplitude-damping channel (ADC) are the following:

$$E_0(q_i) = |0\rangle\langle 0| + \sqrt{p_i}|1\rangle\langle 1|, \quad E_1(q_i) = \sqrt{q_i}|0\rangle\langle 1|,$$

where $q_i \equiv 1 - p_i$ is the amplitude-damping coefficient.

State	α	p	C	N	E_R	B
ρ_0	0.5000	1.0000	1.0000	1.0000	1.0000	1.0000
ρ_1	0.0369	1.0000	0.3770	0.3770	0.2279	0.3770
ρ_2	0.0751	1.0000	0.5271	0.5271	0.3847	0.5271
ρ_3	0.2198	0.8536	0.7070	0.5756	0.4039	0.0000
ρ_4	0.3510	0.9565	0.9130	0.8706	0.7445	0.8169
ρ_5	0.2116	1.0000	0.8169	0.8169	0.7445	0.8169
ρ_6	0.0947	1.0000	0.5856	0.5856	0.4520	0.5856

Phase damping channel (PDC)

The Kraus operators for the single-qubit phase-damping channel (PDC) can be given by:

$$E_0(q_i) = |0\rangle\langle 0| + \sqrt{p_i}|1\rangle\langle 1|, \quad E_1(q_i) = \sqrt{q_i}|1\rangle\langle 1|$$

where $q_i = 1 - p_i$ is the phase-damping coefficient.

Maximally-entangled mixed states (A)

In case of one-sided amplitude damping we obtain

$$\rho^{(\mathrm{A})}(\alpha, q_1) = \sum_i [E_i(q_1)\otimes I]\rho_{\mathrm{in}}(\alpha)[E_i^\dagger(q_1)\otimes I].$$

for which the REE reaches the largest value for a given value of nonlocality for the following coefficients:

$$\begin{aligned} p &= \frac{1}{4}\left(2+\sqrt{2+2B^2}\right) && \text{if } B < B_0,\\ 1 &\geq p \geq \frac{1}{4}\left(2+\sqrt{2+2B_0^2}\right) && \text{if } B = B_0,\\ p &= 1 && \text{if } B > B_0,\\ \alpha &= \frac{1}{2p}\left(p-\sqrt{5p^2-4p-B^2}\right), \end{aligned}$$

where $B_0 = 0.81686$. These states satisfy the extremal KKT (KarushKuhn-Tucker) conditions for this nonlinear problem as we show in Ref. [6].

Horodecki states (H)

The amplitude-damped state, a mixture of a Bell state $|\psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$ and vacuum, i.e.,

$$\rho^{(\mathrm{H})}(p) = \rho^{(\mathrm{A})}(1/2, p) = p|\psi^+\rangle\langle\psi^+| + (1-p)|00\rangle\langle 00|,$$

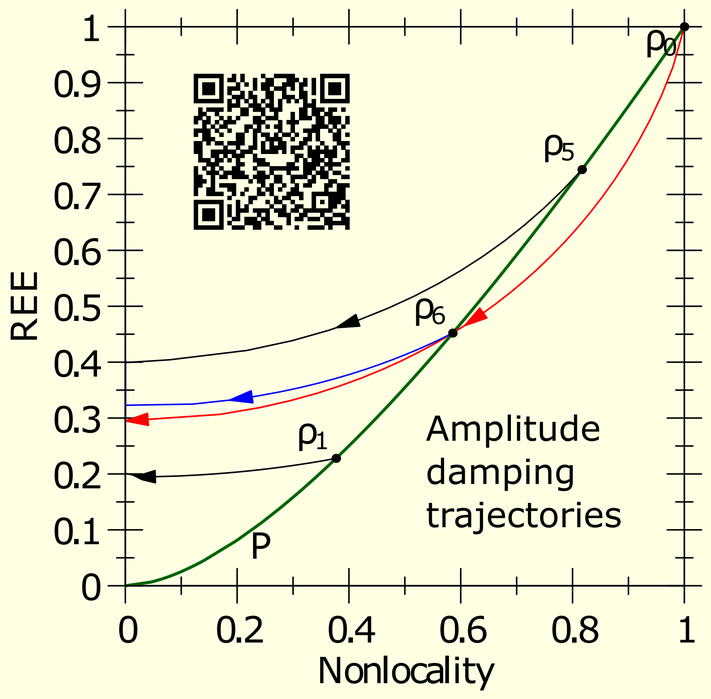
is referred to as the Horodecki state.

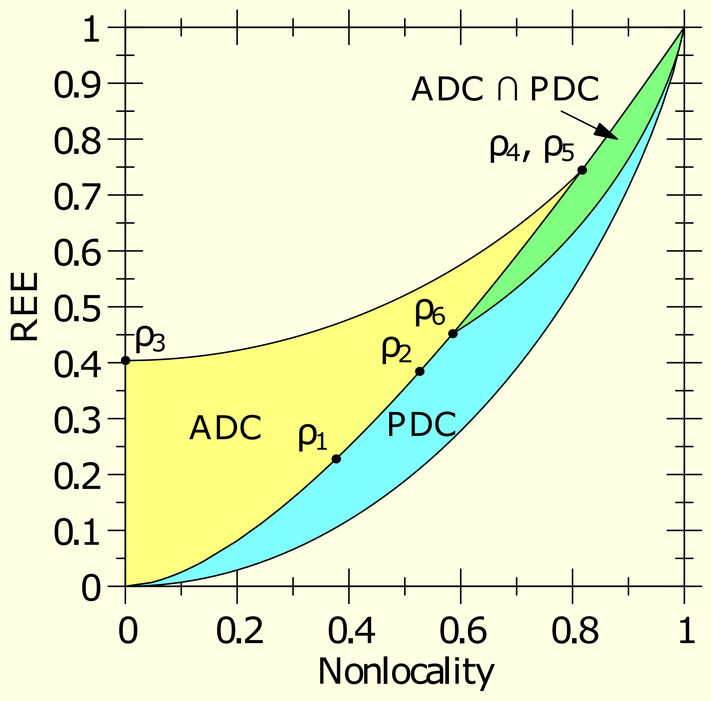
Bell-diagonal states (D)

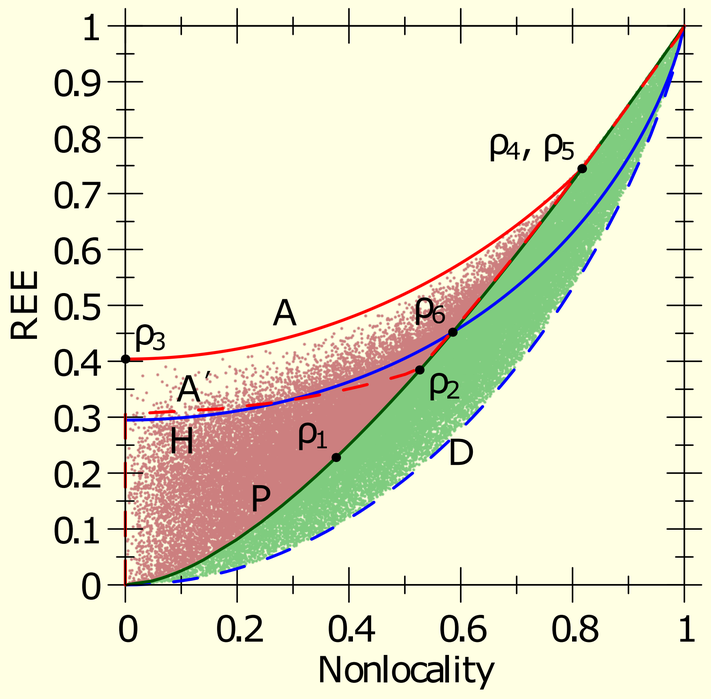
The Bell-diagonal states (labeled by D) can are defined by

$$\rho^{(\mathrm{D})} = \sum_{i=1}^{4}\lambda_i|\beta_i\rangle\langle\beta_i|,$$

where $|\beta_i\rangle$ are the Bell states and $0 < \lambda_j < 1$ such that $\sum_j \lambda_j = 1$ with the largest eigenvalue $\max_j \lambda_j \equiv (1+N)/2 \geq 1/2$. The lower bounds of the REE vs negativity and the REE vs nonlocality correspond to the (rank-2) Bell-diagonal state $\rho^{(\mathrm{D})}$, which satisfy the extremal KKT (KarushKuhnTucker) conditions as we show [6]. This state can be obtained by phase-damping of one of the Bell states. This result could have been aticipated as discussed in Ref. [8].







States maximizing REE for fixed N (A')

The states that maximize REE for a fixed value of negativity are amplitude-damped states described in Ref. [7].

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