Two-qubit mixed states more entangled than pure states: Comparison of the relative entropy of entanglement for a given nonlocality

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In collaboration with



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B. Horst, K. Bartkiewicz, and A. Miranowicz, Phys. Rev. A **87**, 042108 (2013), arXiv:1301.2969

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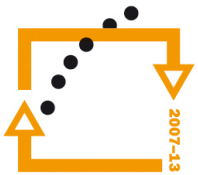

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Outline of the presentation

1. Introduction
   - Bell inequalities
   - Quantum entanglement
   - Kraus operator framework
2. Extremal entanglement for mixed sates
   - Mixed states obtained by damping
   - Conditional extreme entanglement
3. Conclusions

Bloch representation

The two-qubit density matrix in the Bloch representation reads

$$\rho = \frac{1}{4}\Big(I_d \otimes I_d + \sum_{i=1}^{3} x_i \sigma_i \otimes I_d + \sum_{i=1}^{3} y_i I_d \otimes \sigma_i + \sum_{i,j=1}^{3} T_{ij} \sigma_i \otimes \sigma_j\Big),$$

where x_i and y_i represent the Bloch vectors for the two qubits, and $T_{ij} = \mathrm{Tr}(\rho(\sigma_i \otimes \sigma_j))$ are the components of the correlation matrix.

Bell inequalities



J. F. Clauser, *et al.*, Phys. Lett. A **23**, 880 (1969).

Bell-CHSH inequality

$$\langle B_{\mathrm{CHSH}} \rangle = 2\sqrt{\max(0, M)} < 2,$$

where $M = \sum_{n=1}^{3} t_n - \min_{n=1,2,3}(t_n)$ for $t_n = \mathrm{eig}(TT^T)_n$.



R. Horodecki *et al.*, Phys. Lett. A 200, 340 (1995).

B – a measure of nonlocality

$$B(\rho) \equiv \sqrt{\max\left[0,\ M(\rho) - 1\right]}.$$

$B(\rho) = 0$ if a given state ρ satisfies the Bell-CHSH inequality, and $B(\rho) = 1$ if the inequality is maximally violated.

Bell inequalities



C. Śliwa, Phys. Lett. A **317**, 165 (2003);
D. Collins and N. Gisin, J. Phys. A: Math. Gen. **37**, 1775 (2004).

Śliwa-Collins-Gisin inequality

so-called I_{3322} requires performing dichotomous measurement with three settings per subsystem.



T. Vertesi, Phys. Rev. A **78**, 032112 (2008).

Vertesi inequalities

at least 465 settings per side. The Werner states violate this inequality for $p > 0.7056$ instead of $p > 0.7071$ for CHSH.

Compact formula only for the maximal violation of CHSH inequality.

Entanglement measures



W. K. Wooters, Phys. Rev. Lett. **80**, 2245 (1998).

Concurrence

The concurrence is calculated as

$$C(\rho) = \max\{0, 2\lambda_{\max} - \sum_j \lambda_j\},$$

where $\lambda_j^2 = \mathrm{eig}[\rho(\sigma_2 \otimes \sigma_2)\rho^*(\rho_2 \otimes \sigma_2)]_j$ and $\lambda_{\max} = \max_j \lambda_j$.

Directly related to the entanglement of formation, $E_F(\rho)$.

Entanglement measures



A. Peres, Phys. Rev. Lett. **77**, 1413 (1996);
M. Horodecki *et al.*, Phys. Lett. A **223**, 1 (1996).

Negativity

The the negativity is, defined as

$$N(\rho) = \max\{0, -2\mu_{\min}\},$$

where $\mu_{\min} = \min \operatorname{eig}(\rho^{\Gamma})$ and Γ denotes partial transpose.

The negativity is related to the logarithmic negativity, $\log_2[N(\rho)+1]$.

Entanglement measures



V. Verdal *et al.*, Phys. Rev. Lett. **78**,2275 (1997).

Relative entropy of entanglement (REE)

The REE is defined as

$$E_R(\rho) = \min_{\sigma \in \mathcal{D}} S(\rho||\sigma),$$

which is the relative entropy $S(\rho||\sigma) = \mathrm{Tr}\left(\rho \log_2 \rho - \rho \log_2 \sigma\right)$ minimized over a set $\mathcal{D}$ of separable states σ.

No general compact formula for REE even for the two-qubit case. However, for some cases simple formulas exist.



A. Miranowicz and S. Ishizaka, Phys. Rev. A **78**, 032310 (2008).

Kraus operator framework

We analyze the effect of damping channels on the pure states:

$$|\psi_\alpha\rangle = \sqrt{\alpha}|01\rangle + \sqrt{1-\alpha}|10\rangle, \qquad \text{where} \qquad 0 \leq \alpha \leq 1$$

Damping

Two-side damping of the state $\rho_{\text{in}}(\alpha) = |\psi_\alpha\rangle\langle\psi_\alpha|$ results in

$$\rho(\alpha, q_1, q_2) = \sum_{i,j} [E_i(q_1) \otimes E_j(q_2)] \rho_{\text{in}}(\alpha) [E_i^\dagger(q_1) \otimes E_j^\dagger(q_2)],$$

where the Kraus operators satisfy the normalization relation

$$\sum_i E_i^\dagger(q) E_i(q) = I.$$

Mixed states obtained by damping

Amplitude damping

$$E_0(q_i) = |0\rangle\langle 0| + \sqrt{p_i}|1\rangle\langle 1|, \quad E_1(q_i) = \sqrt{q_i}|0\rangle\langle 1|,$$

where $q_i = 1 - p_i$ is the amplitude-damping coefficient.

Amplitude-damped states

In case of one-sided amplitude damping we obtain:

$$\rho^{(\mathrm{A})}(\alpha, q_1) = \sum_i [E_i(q_1) \otimes I]\rho_{\mathrm{in}}(\alpha)[E_i^\dagger(q_1) \otimes I].$$

This channel has a simple physical interpretation if the qubits are encoded as Fock states.

Mixed states obtained by damping

Phase damping

$$E_0(q_i) = |0\rangle\langle 0| + \sqrt{p_i}|1\rangle\langle 1|, \quad E_1(q_i) = \sqrt{q_i}|1\rangle\langle 1|$$

where $q_i = 1 - p_i$ is the phase-damping coefficient.

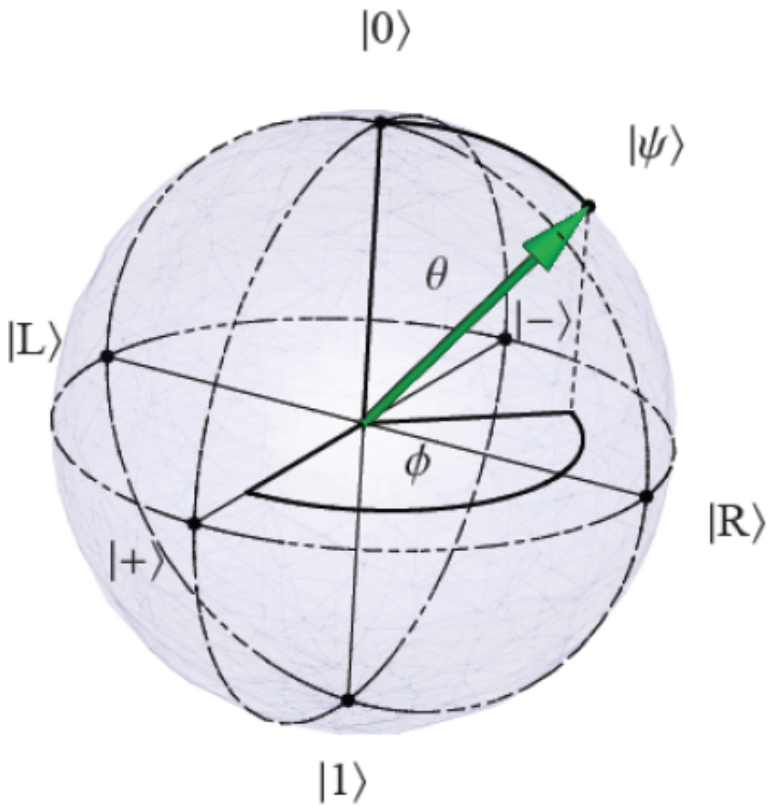
Phase-damped states (Bell diagonal states)

Phase damping of one of the Bell states $|\beta_i\rangle$ results in

$$\rho^{(\mathrm{D})} = \sum_{i=1}^{4} \lambda_i |\beta_i\rangle\langle\beta_i|.$$

where $0 < \lambda_j < 1$ such that $\sum_j \lambda_j = 1$ with the largest eigenvalue $\max_j \lambda_j \equiv (1+N)/2 \geq 1/2$.

Polarization qubits



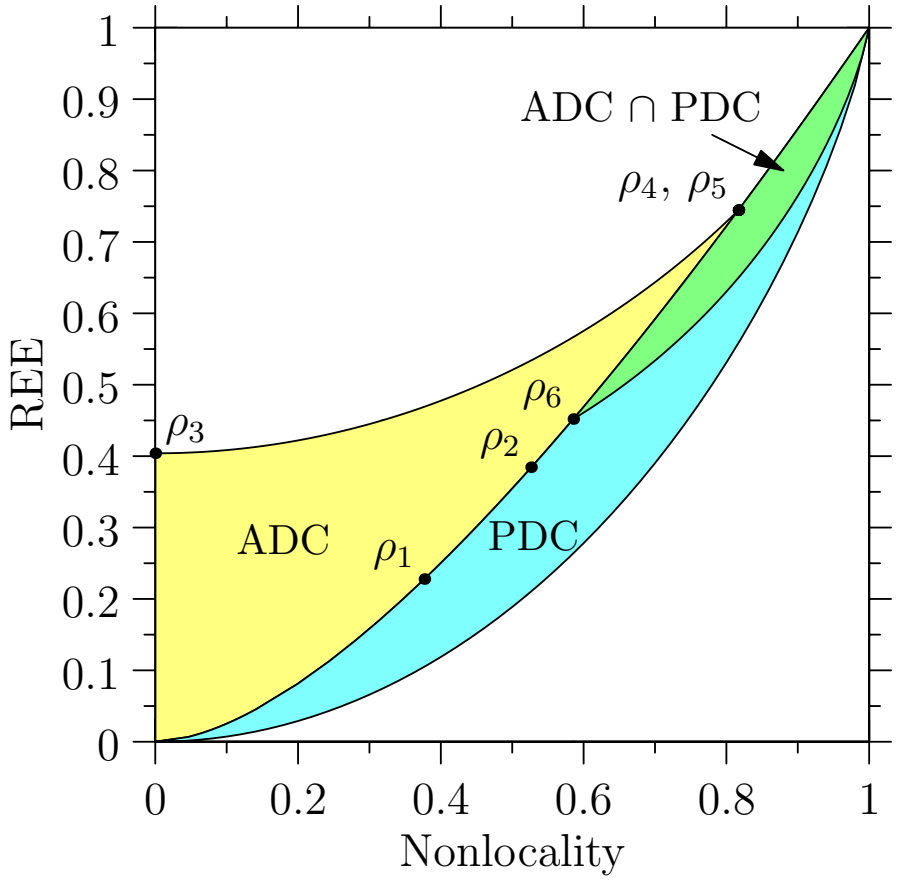
Polarization encoding

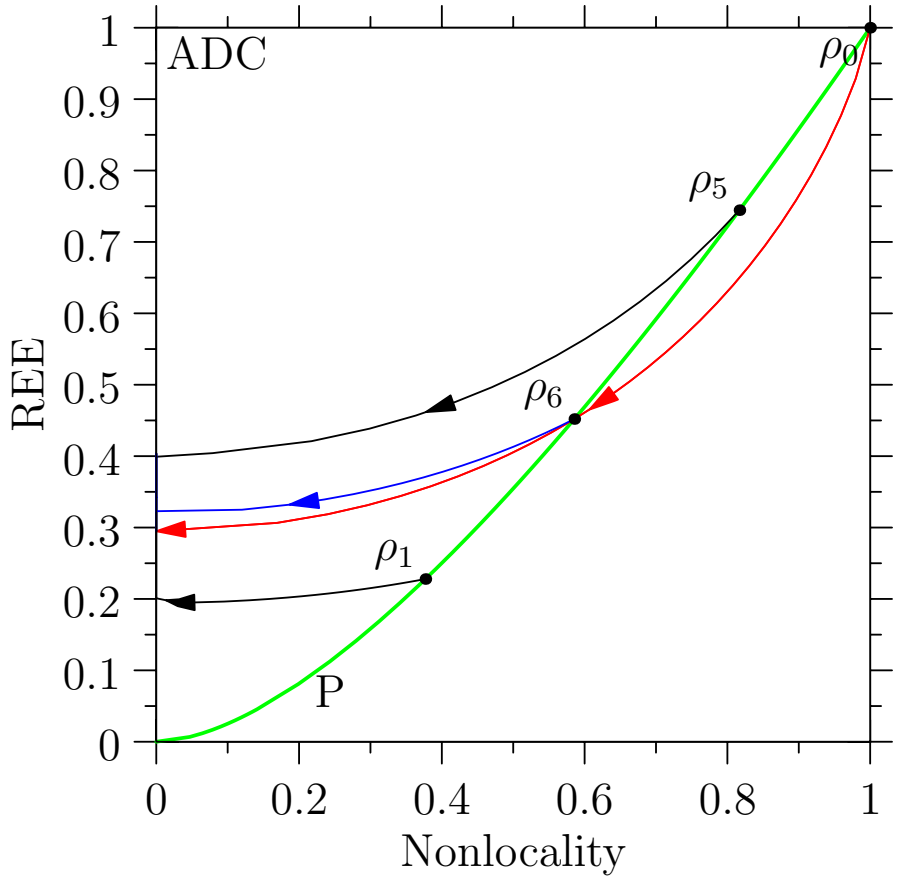
$$|0\rangle_n \equiv |1_H, 0_V\rangle_n = \hat{a}^\dagger_{n,H}|0\rangle$$
$$|1\rangle_n \equiv |0_H, 1_V\rangle_n = \hat{a}^\dagger_{n,V}|0\rangle$$

Bell polarization states

$$|\Phi^\pm\rangle_{m,n} = \frac{1}{\sqrt{2}}(\hat{a}^\dagger_{m,h}\hat{a}^\dagger_{n,h} \pm \hat{a}^\dagger_{m,v}\hat{a}^\dagger_{n,v})|0\rangle, \quad |\Psi^\pm\rangle_{m,n} = \frac{1}{\sqrt{2}}(\hat{a}^\dagger_{m,h}\hat{a}^\dagger_{n,v} \pm \hat{a}^\dagger_{m,v}\hat{a}^\dagger_{n,h})|0\rangle.$$

REE, nonlocality, and damping

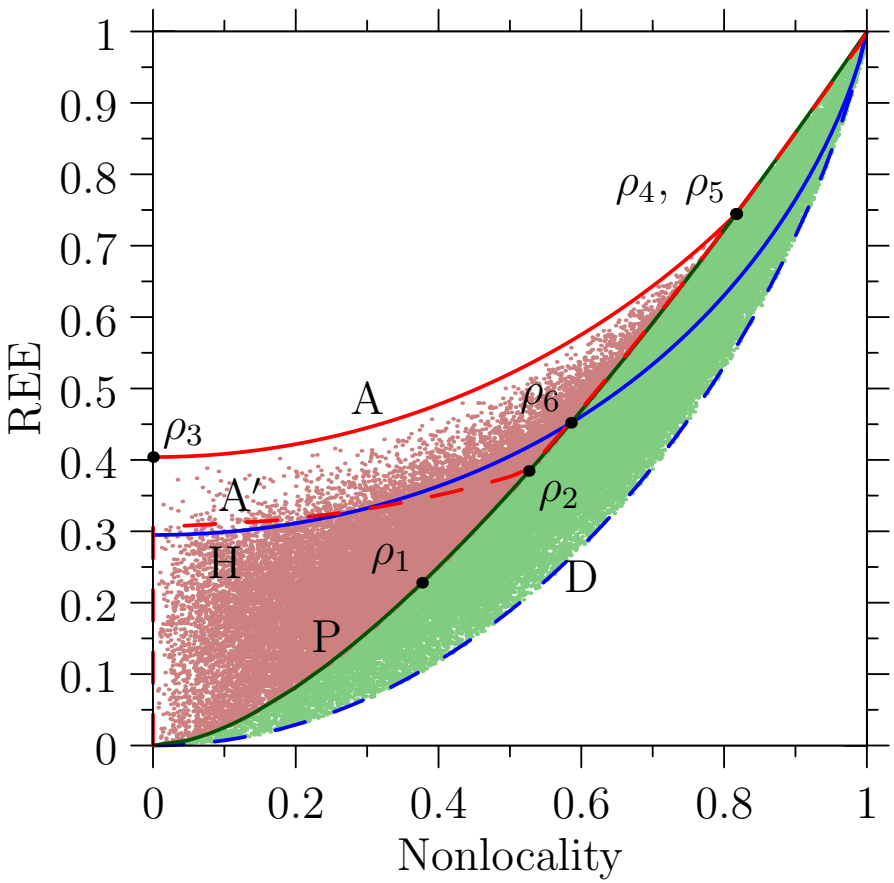


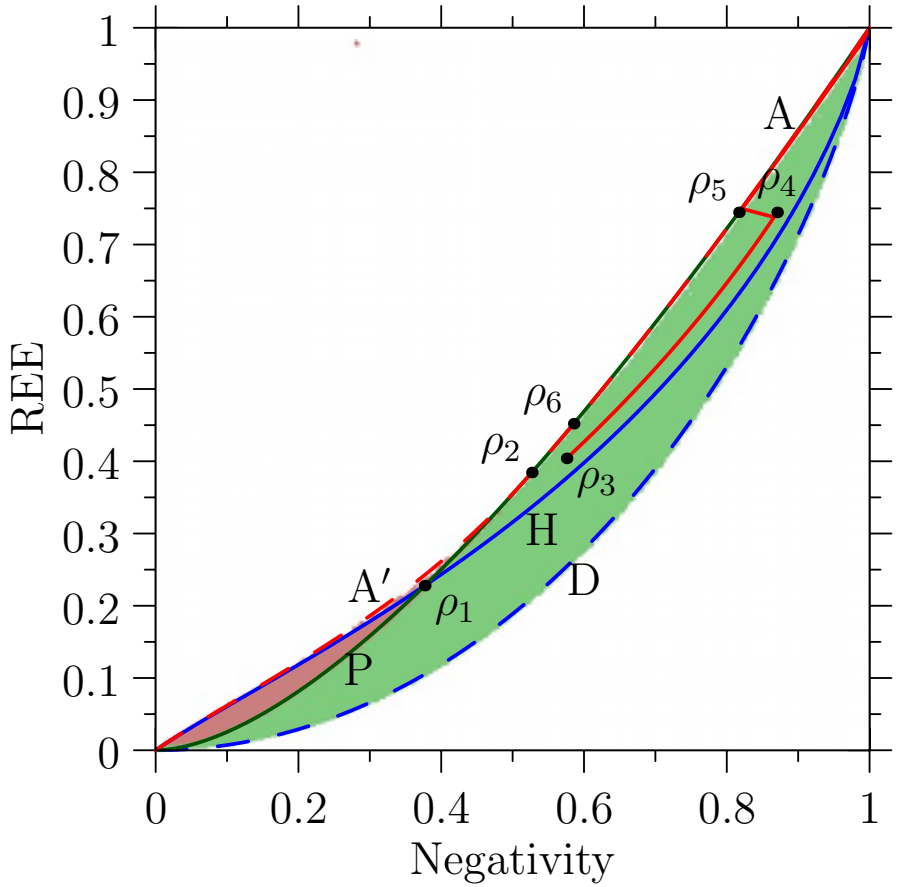


Maximal nonlocality can reached by Bell diagonal states.

F. Verstraete, M. M. Wolf, Phys. Rev. Lett. **89** 170401 (2002).

Mixed states more entangled than pure states





State	α	p	C	N	E_R	B
ρ_1	0.0369	1.0000	0.3770	0.3770	0.2279	0.3770
ρ_2	0.0751	1.0000	0.5271	0.5271	0.3847	0.5271
ρ_3	0.2198	0.8536	0.7070	0.5756	0.4039	0.0000
ρ_4	0.3510	0.9565	0.9130	0.8706	0.7445	0.8169
ρ_5	0.2116	1.0000	0.8169	0.8169	0.7445	0.8169
ρ_6	0.0947	1.0000	0.5856	0.5856	0.4520	0.5856

Extremal entanglement: REE

The Lagrange function

$$\begin{aligned}\mathcal{L} &= \mathrm{Tr}(\rho\log\rho) - \mathrm{Tr}(\rho\log\sigma) + l[\mathrm{Tr}(\rho B_{\mathrm{CHSH}}) - \beta] \\ &\quad - \mathrm{Tr}(X\rho) + \lambda(\mathrm{Tr}\rho - 1),\end{aligned}$$

where l, λ, and X are Lagrange multipliers and $\beta = 2\sqrt{M(\rho)}$

The Karush-Kuhn-Tucker (KKT) conditions

$$\begin{aligned}0 &= \log\rho - \log\sigma + P + l\mathcal{B}_{\mathrm{CHSH}} - X - E_R(\rho) - 1 - l\beta, \\ &\quad X \geq 0, \quad \mathrm{Tr}(X\rho) = 0.,\end{aligned}$$

where E_R is REE defined for $\log$ instead of $\log_2$.

Extremal entanglement: REE

In case of one-sided amplitude damping we obtain

$$\rho^{(\mathrm{A})}(\alpha, 1-p) = \sum_i [E_i(1-p) \otimes I]\rho_{\mathrm{in}}(\alpha)[E_i^\dagger(1-p) \otimes I].$$

Extremal states: maximal REE

$$\begin{aligned} p &= \begin{cases} \frac{1}{4}\left(2+\sqrt{2+2B^2}\right) & \text{if } B < B_0, \\ 1 & \text{if } B > B_0, \end{cases} \\ 1 &\geq p \geq \frac{1}{4}\left(2+\sqrt{2+2B_0^2}\right) \qquad \text{if } B = B_0, \\ \alpha &= \frac{1}{2p}\left(p-\sqrt{5p^2-4p-B^2}\right), \end{aligned}$$

where $B_0 = 0.81686$ and REE is obtained by solving numerically a single-variable equation.

Extremal entanglement: REE

Extremal states: minimal REE

The optimal (rank-2) Bell-diagonal state

$$\rho^{(\mathrm{D})} = p|\psi^+\rangle\langle\psi^+| + (1-p)|\psi^-\rangle\langle\psi^-|,$$

can be created by applying phase-damping to one of the Bell states.

The REE for the extremal states reads

$$E_R(\rho^{(\mathrm{D})}) = 1 - h\Big(\frac{1+N}{2}\Big),$$

where $h = -x\log_2 x - (1-x)\log_2(1-x)$ and $N = C$ is related to p since $\max(p, 1-p) \equiv (1+N)/2 \geq 1/2$.

Conclusions

- We found states of the extremal E_R for a given B. The states satisfy the KKT conditions and provide boundaries for the REE values obtained by Monte Carlo simulations.
- Mixed states can be more entangled in terms of E_R than pure states if the nonlocality $B \in (0, 0.82)$ and the maximal difference between these REEs is $\Delta E_R = 0.4$.
- These mixed states can by obtained by the ordinary amplitude damping of pure states.
- We believe that these results can stimulate further quest for practical protocols of quantum information processing for which mixed states are more effective than pure states.

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Thank you for your attention!



B. Horst, K. Bartkiewicz, and A. Miranowicz, Phys. Rev. A **87**, 042108 (2013), arXiv:1301.2969